Motivation: observed total effect, and the exposure is not intervenable

Notation

- $E[Y_a M_a]$: expectation that takes over both outcome and the random variable M_a (that is, both Y and M_a are RVs), so only intervene on exposure.

1. total effect: $E[Y_{aM_a}] - E[Y_{a*M_a*}] = E[Y_a] - E[Y_a*]$(ATE) notational convention: $Y_a M_a = Y_a$

2. ???controlled direct effect $E[Y_{am}] - E[Y_{a^*m}]$: with mediator controlled at level m (control on m will introduce selection bias, M is the common cause of confounder and A page 7)

why controlled direct effct is of interest

3. natural indirect/direct effect:

$$
\begin{aligned}
E[Y_a] - E[Y_{a^*}] &= E[Y_{aM_a}] - E[Y_{a^*M_{a^*}}] \\
&= \big(E[Y_{aM_a}] - E[Y_{aM_{a^*}}]\big)(1) \\
&+ \big(E[Y_{aM_{a^*}}] - E[Y_{a^*M_{a^*}}]\big)(2)
\end{aligned}
$$

where term (1) is the natural indirect effect (A fixed at A=a, so there is no direct effect); term(2) is the natural direct effect: M_{a^*} are from the same distribution. why called natural?: intervene on the exposure and do not intervene on mediator.

other names:

- $E[Y_{aM_a}] - E[Y_{aM_{a^*}}]$: indirect effect + interaction effect($E[Y_{aM_a}]$ a=1 versus a*=0) (this is also called **total indirect effect)**
- $E[Y_{aM_{a^*}}] - E[Y_{a^*M_{a^*}}]$**: pure direct effect**

Alternative decomposition:

$$
\begin{aligned}
E[Y_a] - E[Y_{a^*}] &= E[Y_a M_a] - E[Y_{a^*} M_{a^*}] \\
&= \big(E[Y_{aM_a}] - E[Y_{a^*M_a}]\big)(\textbf{??natural direct effect}) \\
&+ \big(E[Y_{a^*M_a}] - E[Y_{a^*M_{a^*}}]\big)(\textbf{??natural indirect effect})
\end{aligned}
$$

$E[Y_{aM_a}] - E[Y_{a^*M_a}]$**: direct effect + interaction**

other names:

- $E[Y_{aM_a}] - E[Y_{a^*M_a}]$**: direct effect + interaction effect(**$E[Y_{aM_a}]$ **a=1 versus a*=0) (this is also called total direct effect)**

- $E[Y_{a^*M_a}] - E[Y_{a^*M_{a^*}}]$**: pure indirect effect**

"cross-world": potential outcomes Y_{aMa^*} where the direct pathway is intervened at exposure level a and the indirect pathway at level a

Identification of NIE and NDE

Assumptions

- **1. joint exchangeability:** $Y_{am} \perp\!\!\!\perp (A, M) \mid X$
- **2. positivity :** $P(A = a, M = m|x) > 0$ **for all a, m**
- **3. consistency:** $Y_{AM} = Y$
- **4.** $M_a \perp\!\!\!\perp A \mid X$: **we have to observe all the confounders of the exposure on mediator effect.**
- **5.** $Y_{am} \perp\!\!\!\perp M_{a^*} \mid X$**:cross-world independence assumption**

$$
\begin{aligned}
E(C) &= E_A\left\{ E_{B|A}\left[E_C(C \mid A.B) \right] \right\} \\
&= \int_A E_{B|A}\left[E_C(C \mid AB) \right] f(A)\, dA \\
&= \int_A \int_{B|A} E_C(C \mid AB)\, f(B|A)\, dB\, f(A)\, dA \\
&= \int_A \int_{B|A} \underbrace{\int_C C\, f(C \mid AB)\, dC}_{\text{joint of } AB}\, f(B|A)\, \underbrace{dB}_{B|A}\, f(A)\, dA \\
&= \int_C C \cdot \int_A \int_{B|A} f(C \mid AB) \cdot f(B|A)\, f(A)\, dB\, dA\, dC \\
&= \int_C C\, f(C)\, dC \\
&= E(C)
\end{aligned}
$$

$$
\begin{aligned}
E[Y_{aM_{a^*}}] &= E_X\left(E_{M_{a^*}|X} E[Y_{aM_{a^*}} \mid M_{a^*}, X]\right) \textbf{ law of total expectation} \\
&= \sum_x \sum_m E[Y_{am} \mid x]^{\textbf{cross-world indep assumption}} P(M_{a^*} = m \mid x)^{\textbf{mediator dist}} P(X = x)^{\textbf{confounder}} \\
&= \sum_x \sum_m E[Y^{\textbf{A3consistency}} \mid A = a, M = m^{\textbf{A1}}, x] P(M = m \mid A = a^{*\textbf{A4}}, x) P(X = x).
\end{aligned}
$$